Two oscillators quantum groups and associated deformed coherent states

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Abstract

Starting from a faithful five-dimensional matrix representation of the group of two independent oscillators and applying the R-matrix method we generate some classes of deformed fermionic-bosonic quantum Hopf algebras. The corresponding Lie deformed superalgebras of type I–II, obtained by duality, are computed and a realization of generators of these deformed superalgebras are given in terms of the usual fermionic and bosonic creation and annihilation operators associated to the supersymmetric harmonic oscillator. Then, a generalized deformed annihilator is construted and their eigenstates are computed giving a new class of deformed coherent states.

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1 Introduction

Quantum groups [1, 2, 3] are Hopf algebras equiped with a quasitriangular structure. They are generalizations of Lie groups and algebras. Since their creation in the mid-eighteen, quantum groups have attracted considerable attention in theoretical and mathematical physics because the richness of their algebraic structure reflected in useful technical elements such as coproduct, twisted product and counit, linked to many potential applications. For example, in relation with these three properties it have been established that quantum field are an example of infinite-dimensional quantum groups [4]. In general, the concept of deformed quantum Lie algebras has found various applications in quantum optics, quantum field theory, quantum statistical mechanics, supersymmetric quantum mechanics and some purely mathematical problems. For instance, in the case of boson quantum algebras, the special coproduct properties are useful to characterize multi-particle Hamiltonians [5]. Recently, deformed coherent and squeezed states have been associated to quantum Heisenberg algebras [6]. In the case of the Poincaré quantum algebra, the coproduct have been brought to bear the study the fusion of phonons [7]. In the case of the $su_q(2)$ algebra, it has been found that the $su_q(2)$ effective Hamiltonians reproduce accurately the physical properties of the $su(2) \oplus h(2)$ models [8].

The goal of this article is firstly, to use the R–matrix method [1, 2, 9] to generate some deformations of the group of two independent oscillators and the corresponding deformed quantum Lie algebras. More precisely, we consider a faithful five-dimensional matrix representation of this group and we apply the R–matrix method to compute the deformed quantum groups and quantum Lie superalgebras of fermionic-bosonic type. Next, we will find a representation of these deformed algebras and construct some classes of deformed coherent states.

We recall that, the R–matrix method has been used to generate quadratic relations between the basic elements of a given group, considered as generators of a bialgebra with coproduct derived from the group law. In this way, deformed bialgebras have been defined whose external consistency is ensured if R satisfies the well-known quantum Yang-Baxter equation (QYBE). The associated deformed quantum Lie algebras (or superalgebras) have been obtained by defining a suitable duality operation.

This method has been applied, for instance, to the $GL_q(2,\mathbb{C})$ and $GL_{p,q}(2,\mathbb{C})$ ma-

trix quantum groups ([10] and [11], respectively), the $GL(2, \mathbb{C})$ matrix group [12], the $GL(1\|1)$ supergroup [13], Heisenberg group [14] and oscillator group [15]. In the case of the Heisenberg group, the only deformed Lie algebras that can be obtained are of the bosonic type whereas in the cases of the $GL(2, \mathbb{C})$ and oscillator groups these are of the fermionic and bosonic type.

Considering the case of the group of two independent oscillators, three types of deformed Lie algebras appear, i.e., fermionic-fermionic, fermionic-bosonic and bosonic-bosonic ones. Although these three types of deformed algebras are interesting, we pay special attention to the fermionic-bosonic type because these algebras contain as subalgebras the deformed Heisenberg-Weyl Lie superalgebra which introduces new features concerning the study of deformed coherent states.

This article is organized as follows. In section 2, we give a description of the R-matrix method by applying it to a five-dimensional faithful matrix representation of the group of two independent oscillators. In section 3, we compute the deformed fermionic-bosonic bialgebras of the type I-II and the corresponding deformed Lie superalgebras associated to this group. In section 4, we give some realizations of this deformed superalgebras in terms of the usual creation and annihilation operators associated to the standard and supersymmetric harmonic oscillators. In section 5, based on the preceding realizations, we define the deformed supersymmetric and generalized harmonic oscillator annihilators, and we compute their eigenstates. We interpret these eigenstates as coherent states associated to the two oscillator quantum group. Details of some calculus are given in Appendices A and B.

2 Two oscillator quantum groups

In this section we describe the R-matrix method [9] to construct certain classes of deformed quantum algebras associated to the direct product of two oscillator groups.

The starting point is a five-dimensional faithful matrix representation of the group

of two independent oscillators. An element is given by

$$T = \begin{pmatrix} 1 & \alpha & \beta & 0 & 0 \\ 0 & \eta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & c & d & 0 \\ 0 & 0 & b & a & 1 \end{pmatrix}, \tag{2.1}$$

where the parameters $\alpha, \beta, \gamma, \eta, a, b, c, d$ and the unity 1 are considered as the generators of a commutative algebra $\mathcal{A}$, the space of linear functions of these generators, provided with a structure of Hopf algebra by tensorial multiplication:

$$\begin{aligned} \Delta 1 &= 1 \otimes 1, \\ \Delta\alpha &= 1 \otimes \alpha + \alpha \otimes \eta, \qquad \Delta\beta = 1 \otimes \beta + \beta \otimes 1 + \alpha \otimes \gamma, \\ \Delta\gamma &= \eta \otimes \gamma + \gamma \otimes 1, \qquad \Delta\eta = \eta \otimes \eta, \\ \Delta a &= 1 \otimes a + a \otimes d, \qquad \Delta b = 1 \otimes b + b \otimes 1 + a \otimes c, \\ \Delta c &= d \otimes c + c \otimes 1, \qquad \Delta d = d \otimes d. \end{aligned} \tag{2.2}$$

According to this co-multiplication law, the generators of the corresponding Lie algebra, in the representation space $\mathcal{A}$, are given by

$$X_1 = \frac{\partial}{\partial\alpha}, \qquad X_2 = \frac{\partial}{\partial\beta}, \tag{2.3}$$

$$X_3 = \alpha\frac{\partial}{\partial\beta} + \eta\frac{\partial}{\partial\gamma}, \qquad X_4 = \eta\frac{\partial}{\partial\eta} + \alpha\frac{\partial}{\partial\alpha}, \tag{2.4}$$

$$\tilde{X}_1 = \frac{\partial}{\partial a}, \qquad \tilde{X}_2 = \frac{\partial}{\partial b}, \tag{2.5}$$

$$\tilde{X}_3 = a\frac{\partial}{\partial b} + d\frac{\partial}{\partial c}, \qquad \tilde{X}_4 = d\frac{\partial}{\partial d} + a\frac{\partial}{\partial a}. \tag{2.6}$$

They verify the commutation relations of the Lie algebra $ho(4,\mathbb{R}) \oplus ho(4,\mathbb{R})$, constructed as the direct sum of the well-known Lie algebras associated with two independent harmonic oscillators . The non-zero commutation relations of this Lie algebra are given by

$$[X_1, X_3] = X_2, \qquad [X_4, X_1] = -X_1, \qquad [X_4, X_3] = X_3, \tag{2.7}$$

$$\left[\tilde{X}_1, \tilde{X}_3\right] = \tilde{X}_2, \qquad [\tilde{X}_4, \tilde{X}_1] = -\tilde{X}_1, \qquad [\tilde{X}_4, \tilde{X}_3] = \tilde{X}_3. \tag{2.8}$$

The non-deformed quantum Lie algebra associated to the two oscillator group corresponds to the dual space of $\mathcal{A}$. The action of their generators, $A, B, C, H, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{H}$ on

the generating elements $\mathcal{P} = \beta^k \eta^l \alpha^m \gamma^n b^r a^s d^t c^u,\ k, l, m, n, r, s, t, u \in \mathbb{Z}_+$, of $\mathcal{A}$, is given by [10]

$$\begin{aligned}
(A, \mathcal{P}) &= \Big(X_1 \mathcal{P}\Big)_{\{0\}} = \delta_{k0}\, \delta_{m1}\, \delta_{n0} \delta_{r0}\, \delta_{s0}\, \delta_{u0}, && (2.9)\\
(B, \mathcal{P}) &= \Big(X_2 \mathcal{P}\Big)_{\{0\}} = \delta_{k1}\, \delta_{m0}\, \delta_{n0}\, \delta_{r0}\, \delta_{s0}\, \delta_{u0}, && (2.10)\\
(C, \mathcal{P}) &= \Big(X_3 \mathcal{P}\Big)_{\{0\}} = \delta_{k0}\, \delta_{m0}\, \delta_{n1}\, \delta_{r0}\, \delta_{s0}\, \delta_{u0}, && (2.11)\\
(H, \mathcal{P}) &= \Big(X_4 \mathcal{P}\Big)_{\{0\}} = l\, \delta_{k0}\, \delta_{m0}\, \delta_{n0}\, \delta_{r0}\, \delta_{s0}\, \delta_{u0}, && (2.12)\\
(\tilde{A}, \mathcal{P}) &= \Big(\tilde{X}_1 \mathcal{P}\Big)_{\{0\}} = \delta_{k0}\, \delta_{m0}\, \delta_{n0}\, \delta_{r0}\, \delta_{s1}\, \delta_{u0}, && (2.13)\\
(\tilde{B}, \mathcal{P}) &= \Big(\tilde{X}_2 \mathcal{P}\Big)_{\{0\}} = \delta_{k0}\, \delta_{m0}\, \delta_{n0}\, \delta_{r1}\, \delta_{s0}\, \delta_{u0}, && (2.14)\\
(\tilde{C}, \mathcal{P}) &= \Big(\tilde{X}_3 \mathcal{P}\Big)_{\{0\}} = \delta_{k0}\, \delta_{m0}\, \delta_{n0}\, \delta_{r0}\, \delta_{s0}\, \delta_{u1}, && (2.15)\\
(\tilde{H}, \mathcal{P}) &= \Big(\tilde{X}_4 \mathcal{P}\Big)_{\{0\}} = t\, \delta_{k0}\, \delta_{m0}\, \delta_{n0}\, \delta_{r0}\, \delta_{s0}\, \delta_{u0}, && (2.16)
\end{aligned}$$

where $\{0\} \equiv \{\alpha = \beta = \gamma = 0, \eta = 1; a = b = c = 0, d = 1\}$. The action of the product of two of these generators, on an arbitrary element $\mathcal{P}$, is computed with the help of the homomorphism property of the co-multiplication. That is, for generic generators V and W, we get

$$(VW, \mathcal{P}) = (V \otimes W, \Delta\mathcal{P}) = \sum_{(c)} (V \otimes W, \mathcal{P}^{(c)} \otimes \mathcal{P}^{(c)}) = \sum_{(c)} (V, \mathcal{P}^{(c)})(W, \mathcal{P}^{(c)}), \qquad (2.17)$$

where, in the usual notation, $\mathcal{P}^{(c)}$ represents the generic elements of $\mathcal{A}$ generated on the co-multiplication action. Then, the commutator of this generators is computed from

$$([V, W], \mathcal{P}) = (V \otimes W - W \otimes V, \Delta\mathcal{P}). \qquad (2.18)$$

Following this procedure it is easy to show that the generators of the dual space of $\mathcal{A}$ satisfy the same non-zero commutation relations as (2.7-2.8), i.e.,

$$\begin{aligned}
&[A, C] = B, \qquad [H, A] = -A, \qquad [H, C] = C, && (2.19)\\
&\Big[\tilde{A}, \tilde{C}\Big] = \tilde{B}, \qquad [\tilde{H}, \tilde{A}] = -\tilde{A}, \qquad [\tilde{H}, \tilde{C}] = \tilde{C}. && (2.20)
\end{aligned}$$

2.1 The R-matrix method

To obtain the possible associated deformed Lie algebras (superalgebras), we apply the well-known R–matrix method[2]. Firstly, we deform the basic algebra $\mathcal{A}$ with help of

a matrix R which satisfies the quantum Yang-Baxter equation(QYBE). Then the corresponding deformed quantum algebras (superalgebras) are obtained by duality, according to the relations (2.9–2.18) (in the case of a superalgebra we must replace in (2.18) the commutator by an anti-commutator and the sign $-$ by a sign $+$ when both V and W represents generators of the type fermionic).

Let us introduce a 25×25 matrix R defined by the equation

$$RT_1T_2 = T_2T_1R, \tag{2.21}$$

where

$$T_1 = T \otimes I, \qquad T_2 = I \otimes T, \tag{2.22}$$

with I the 5×5 identity matrix, and satisfying the QYBE

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{2.23}$$

where

$$R_{12} = R \otimes I, \qquad R_{23} = I \otimes R \tag{2.24}$$

and R_{13} have a similar expression. Writing R in the form of block matrices formed by the 5×5 dimensional sub-matrices R^{ij}, $i, j = 1, 2 \ldots, 5$ and denoting their entries by $(R^{ij})_{kl} = r^{ij}_{kl}$, $k, l = 1, 2, \ldots, 5$, and the entries of T by t_{ij}, we can write explicitly the deformed law (2.21) as

$$r^{ij}_{kl}\, t_{jm}\, t_{ls} \;=\; t_{kl}\, t_{ij}\, r^{jm}_{ls}, \tag{2.25}$$

with summation over repeated indices. In the same way, the QYBE (2.23) writes

$$r^{ij}_{kl}\, r^{lm}_{np}\, r^{jq}_{sn} \;=\; r^{kl}_{sn}\, r^{ij}_{np}\, r^{jq}_{lm}. \tag{2.26}$$

2.2 Commutative bialgebra

In the particular case where R is the identity matrix, $r^{ij}_{kl} = \delta^{ij}\delta_{kl}$, equation (2.25) becomes

$$t_{im}t_{ks} = t_{ks}t_{im}, \quad \forall\, i, m, k, s = 1, 2, \ldots, 5, \tag{2.27}$$

i.e., we regain the commutative algebra $\mathcal{A}$.

2.3 Non-commutative bialgebra

Other diagonal R–matrix compatible with the coproduct (2.2) for which we get a non-commutative (non deformed) bialgebra is given by the fermionic–bosonic type R–matrix, $R^{ij} = \delta^{ij}\tilde{I}$ where $\tilde{I} = I$ when $(i,j) \neq (2,2)$ and $\tilde{I} = \mathrm{diag}\,(1,-1,1,1,1)$ when $(i,j) = (2,2)$. In this case (2.25) implies

$$\{\alpha,\eta\} = 0, \qquad \{\gamma,\eta\} = 0, \qquad \alpha^2 = 0, \qquad \gamma^2 = 0, \tag{2.28}$$

$$[\alpha,\beta] = 0, \qquad [\alpha,\gamma] = 0, \qquad [\beta,\gamma] = 0 \qquad [\beta,\eta] = 0 \tag{2.29}$$

$$[a,b] = 0, \quad [a,c] = 0, \quad [a,d] = 0, \quad [b,c] = 0, \quad [b,d] = 0, \quad [c,d] = 0, \tag{2.30}$$

and all the commutators between the Greek generators and the Roman generators being equal to zero. Here $\{,\}$ denotes the anti-commutator.

In this last case, the non-zero super-commutation relations satisfies by the dual generators are given by

$$\{A,C\} = B, \qquad [H,A] = -A, \qquad [H,C] = C, \tag{2.31}$$

$$[\tilde{A},\tilde{C}] = \tilde{B}, \qquad [\tilde{H},\tilde{A}] = -\tilde{A}, \qquad [\tilde{D},\tilde{C}] = \tilde{C}, \tag{2.32}$$

with

$$A^2 = 0, \qquad C^2 = 0. \tag{2.33}$$

These are the commutation relations of a Lie superalgebra isomorphic to $ho(2/2,\mathbb{R}) \oplus ho(4,\mathbb{R})$.

3 Deformed fermionic-bosonic quantum superalgebras

We are interested to deform the superalgebra $ho(2/2,\mathbb{R}) \oplus ho(4,\mathbb{R})$. To do it, firstly we find a set of deformed R matrices, continuously connected to the non deformed fermionic-bosonic type R-matrix and solving (2.21), then we demand that they verify the QYBE (2.23). In Appendix A, we get a set of this type of deformed matrices and the relations of consistency between their entries. We notice that there are several possible choices for the remaining entries of these R matrices so that they verify the QYBE. These different choices determinate the different types of deformation of the fermionic-bosonic bialgebra (2.28-2.30). Indeed, they enter in the classification given in [15], but with a highest

number of deformation parameters, and we get the type I–I, I–II, I–III II–I II-II and II–III fermionic–bosonic bialgebras. In the following, we give some examples of deformed bialgebras of the I-II fermionic–bosonic type and construct the corresponding quantum superalgebras.

3.1 Type I-II fermionic-bosonic bialgebras

The type I-II fermionic-bosonic quantum groups are obtained by setting $r^{12}_{22} = r^{22}_{23} = 0$ and $r^{54}_{44} = r^{44}_{43} = 0$ in (A.45) and (A.49), respectively. Again, there are several possibilities to choose the R-matrix verifying the QYBE (2.23):

3.1.1 Direct sum bialgebra structure

For instance, the choice ($\epsilon = \pm 1$)

$$r^{12}_{12} = z, \qquad r^{23}_{23} = x, \tag{3.1}$$

$$r^{12}_{23} = r^{23}_{12} = \epsilon\sqrt{zx}, \qquad r^{13}_{22} = r^{13}_{11} - \epsilon\sqrt{zx}, \qquad r^{22}_{13} = r^{11}_{13} - \epsilon\sqrt{zx}, \tag{3.2}$$

$$r^{43}_{14} = r^{53}_{15} = w, \qquad r^{53}_{43} = -r^{43}_{53} = \rho, \qquad r^{44}_{13} = r^{55}_{13} = -w + r^{11}_{13}, \tag{3.3}$$

$$r^{53}_{44} = -r^{44}_{53} = q, \qquad r^{54}_{53} == -r^{53}_{54} = \tau, \qquad r^{53}_{55} = -r^{55}_{53} = p + q, \tag{3.4}$$

$$r^{13}_{33} = r^{13}_{44} = r^{13}_{55} = r^{13}_{11}, \qquad r^{33}_{13} = r^{11}_{13}, \qquad r^{53}_{13} = -r^{13}_{53}, \tag{3.5}$$

with arbitrary real parameters x, z, p, q, w and coefficients $r^{11}_{13}, r^{13}_{11}, r^{13}_{13}, r^{13}_{53}$, produces the deformed quantum algebra for which the non zero commutation relations are given by

$$\{\alpha, \eta\} = 0, \qquad \{\gamma, \eta\} = 0, \qquad \alpha^2 = \frac{1}{2}z(1 - \eta^2), \qquad \gamma^2 = \frac{1}{2}x(1 - \eta^2) \tag{3.6}$$

$$[\alpha, \beta] = z\gamma\eta, \qquad [\gamma, \beta] = x\alpha\eta \tag{3.7}$$

and

$$[a, b] = pa + \tau(1 - d), \qquad [b, c] = -qc - \rho(1 - d), \tag{3.8}$$

i.e., the direct sum of the deformed type I fermionic and type II bosonic quantum groups [15].

The same deformation relations (3.6-3.8), but with $q = -p$, is obtained if we take

$$r^{12}_{12} = z, \qquad r^{23}_{23} = x, \tag{3.9}$$

$$r^{12}_{23} = -r^{23}_{12} = \epsilon\sqrt{zx}, \qquad r^{13}_{22} = -r^{22}_{13} = -r^{11}_{13} + \epsilon\sqrt{zx}, \tag{3.10}$$

$$r_{11}^{13} = -r_{13}^{11}, \qquad r_{13}^{13} = \frac{zx}{2} - \left(r_{13}^{11} - \epsilon\sqrt{zx}\right)^2, \tag{3.11}$$

$$r_{43}^{53} = -r_{53}^{43} = \rho, \qquad r_{44}^{53} = -r_{53}^{44} = -p, \qquad r_{53}^{54} = -r_{54}^{53} = \tau, \tag{3.12}$$

$$r_{33}^{13} = r_{44}^{13} = r_{55}^{13} = -r_{13}^{44} = -r_{13}^{55} = -r_{13}^{33} = -r_{13}^{11} + 2\epsilon\sqrt{zx}, \qquad r_{13}^{53} = -r_{53}^{13}, \tag{3.13}$$

with arbitrary real parameters x, z, p and coefficients r_{13}^{11}, r_{53}^{13}.

3.1.2 Non-direct sum bialgebra structure

Other class of deformed quantum groups that do not show a direct sum structure is given by the relations (3.6-3.8), by taking $q = -p$, $\rho = \tau = 0$ in addition to the commutation relations

$$[\beta, a] = \theta a, \qquad [\beta, c] = -\theta c, \tag{3.14}$$

where θ is an additional arbitrary parameter. Here the non zero matrix elements of R are given by

$$r_{12}^{12} = z, \qquad r_{23}^{23} = x, \tag{3.15}$$

$$r_{23}^{12} = r_{12}^{23} = \epsilon\sqrt{zx}, \qquad r_{13}^{22} = r_{13}^{11} - \epsilon\sqrt{zx}, \qquad r_{22}^{13} = r_{11}^{13} - \epsilon\sqrt{zx}, \tag{3.16}$$

$$r_{13}^{44} = r_{13}^{11} - \theta, \qquad r_{44}^{13} = r_{11}^{13} + \theta, \tag{3.17}$$

$$r_{33}^{13} = r_{55}^{13} = r_{11}^{13}, \qquad r_{13}^{33} = r_{13}^{55} = r_{13}^{11}, \tag{3.18}$$

$$r_{44}^{53} = -r_{53}^{44} = -p, \tag{3.19}$$

with arbitrary real parameters x, z, p, θ and coefficients $r_{13}^{11}, r_{11}^{13}, r_{13}^{13}, r_{53}^{53}$.

A more general quantum group can be reached if we take

$$r_{12}^{12} = z, \qquad r_{23}^{23} = x, \tag{3.20}$$

$$r_{23}^{12} = -r_{12}^{23} = \epsilon\sqrt{zx}, \qquad r_{13}^{13} = \frac{xz}{2} - \left(r_{13}^{11} - \epsilon\sqrt{zx}\right)^2, \tag{3.21}$$

$$r_{22}^{13} = -r_{13}^{22} = \epsilon\sqrt{zx} - r_{13}^{11}, \qquad r_{33}^{13} = r_{55}^{13} = -r_{13}^{33} = -r_{13}^{55} = 2\epsilon\sqrt{zx} - r_{13}^{11}, \tag{3.22}$$

$$r_{11}^{13} = -r_{13}^{11}, \qquad r_{13}^{44} = -r_{44}^{13}, \qquad r_{13}^{53} = -r_{53}^{13}, \tag{3.23}$$

$$r_{43}^{53} = -r_{53}^{43} = \rho, \qquad r_{53}^{54} = -r_{54}^{53} = \tau, \qquad r_{44}^{53} = -r_{53}^{44} = q, \tag{3.24}$$

$$r_{44}^{13} = \sigma - r_{13}^{11} + 2\epsilon\sqrt{zx}, \qquad r_{43}^{13} = -r_{13}^{43} = \frac{\rho\sigma}{q}, \qquad r_{54}^{13} = -r_{13}^{54} = -\frac{\tau\sigma}{p}, \tag{3.25}$$

$$r_{51}^{13} = r_{52}^{23} = r_{53}^{33} = r_{54}^{43} = -r_{13}^{51} = -r_{23}^{52} = -r_{33}^{53} = -r_{43}^{54} = p - q, \tag{3.26}$$

for arbitrary real parameters $x, z, p, q, \rho, \tau, \sigma$ and coefficients r^{11}_{13}, r^{13}_{53}. In this case, the corresponding deformed quantum group is given by (3.6-3.8), in addition to the relations

$$[\beta, a] = \frac{\tau\sigma}{p}(d-1) + \sigma a, \qquad [\beta, c] = \frac{\rho\sigma}{q}(d-1) - \sigma c. \tag{3.27}$$

Let us notice that this results represent new classes of deformed bialgebras having the structure which can not be constructed by considering only the direct sum between the corresponding type I fermionic and a type II bosonic oscillator bialgebras.

3.2 Deformed fermionic-bosonic quantum Lie superalgebras

In general, according to the coproduct law (2.2) and the duality relations (2.9–2.16), we can show that the type I-II fermionic-bosonic deformed quantum Lie superalgebras have the structure of a direct sum of the type I fermionic and type II bosonic deformed quantum oscillator algebras obtained in [15]. This is true even if the deformed bialgebras do not have the structure of a direct sum, as in (3.6-3.8,3.14) or in (3.6-3.8,3.27). We remark that this last fact should play an important role at the moment to compute the basic coproduct relations to the dual bialgebra level.

Indeed, from (2.17), writing $\mathcal{P} = \mathcal{F}\mathcal{B}$, where $\mathcal{F} = \beta^k \eta^l \alpha^m \gamma^n,\ k, l, \in \mathbb{Z}_+, m, n = 0, 1$ and $\mathcal{B} = b^r a^s d^t c^u,\ r, s, t, u \in \mathbb{Z}_+$, using the homomorphism properties of the coproduct (2.2) and considering its special structure as well as the structure of the bialgebras given in the above subsection, for generic generators V and W, we get

$$\begin{aligned}
(VW, \mathcal{F}\mathcal{B}) &= (V \otimes W, \Delta(\mathcal{F}\mathcal{B})) = (V \otimes W, (\Delta\mathcal{F})(\Delta\mathcal{B})) \\
&= \sum_{(c),(c')} (V \otimes W, (\mathcal{F}_1^{(c)} \otimes \mathcal{F}_2^{(c)})\ (\mathcal{B}_1^{(c')} \otimes \mathcal{B}_2^{(c')})) \\
&= \sum_{(c)\ (c')} (V, \mathcal{F}_1^{(c)} \mathcal{B}_1^{(c')})\ (W, \mathcal{F}_2^{(c)} \mathcal{B}_2^{(c')}).
\end{aligned} \tag{3.28}$$

In particular, if $V, W \in \{A, B, C, H\}$, according to the duality relations (2.9–2.16), we deduce

$$\begin{aligned}
(VW, \mathcal{F}\mathcal{B}) &= \sum_{(c)\ (c')} (V, \mathcal{F}_1^{(c)})\ (\mathcal{B}_1^{(c')})_{\{0\}}\ (W, \mathcal{F}_2^{(c)})\ (\mathcal{B}_2^{(c')})_{\{0\}} \\
&= (1, \mathcal{B}) \sum_{(c)} (V, \mathcal{F}_1^{(c)})\ (W, \mathcal{F}_2^{(c)}) = (VW, \mathcal{F})\ (1, \mathcal{B}).
\end{aligned} \tag{3.29}$$

From this result, we have for example

$$(\{V,W\},\mathcal{F}\mathcal{B}) = (\{V,W\},\mathcal{F})\ (1,\mathcal{B}), \tag{3.30}$$

i.e, only the fermionic type part of the quantum bialgebra is essential to compute the anti-commutator (commutator) for this class of generators .

On the other hand, if $V,W\in\{\tilde{A},\tilde{B},\tilde{C},\tilde{H}\}$, in the same way we can show that

$$([V,W],\mathcal{F}\mathcal{B}) = (1,\mathcal{F})\ ([V,W],\mathcal{B}), \tag{3.31}$$

i.e, only the bosonic type part of the quantum bialgebra is essential to compute the commutator for this class of generators.

Now, if $V\in\{A,B,C,H\}$ and $W\in\{\tilde{A},\tilde{B},\tilde{C},\tilde{H}\}$, then from (3.28), according to the pairings (2.9–2.16), we get

$$\begin{aligned}([V,W],\mathcal{F}\mathcal{B}) &= \sum_{(c)\ (c')}\Big[(V,\mathcal{F}_1^{(c)})\ (\mathcal{B}_1^{(c')})_{\{0\}}\ (\mathcal{F}_2^{(c)})_{\{0\}}\ (W,\mathcal{B}_2^{(c')}) \\ &- (\mathcal{F}_1^{(c)})_{\{0\}}\ (W,\mathcal{B}_1^{(c')})\ (V,\mathcal{F}_2^{(c)})\ (\mathcal{B}_2^{(c')})_{\{0\}}\Big].\end{aligned} \tag{3.32}$$

To know the explicit form of (3.32), let us write [15]

$$\begin{aligned}\Delta(\beta^k\eta^l) &= \Gamma^{k,i,j}_{v,w;v'w'}\ \beta^v\eta^{w+l}\alpha^i\gamma^j\otimes\beta^{v'}\eta^{w'+l}\alpha^i\gamma^j \\ &+ \Delta^{k,i,j}_{v,w;v'w'}\ \beta^v\eta^{w+l}\alpha^i\gamma^j\otimes\beta^{v'}\eta^{w'+l}\alpha^{i+1}\gamma^{j+1},\end{aligned} \tag{3.33}$$

$$\begin{aligned}\Delta(\alpha^m\gamma^n) &= \delta_{m0}\delta_{n0}\ (1\otimes 1)+\delta_{m0}\delta_{n1}\ (\eta\otimes\gamma+\gamma\otimes 1)+\delta_{m1}\delta_{n0}\ (1\otimes\alpha+\alpha\otimes\eta) \\ &+ \delta_{m1}\delta_{n1}\ (\eta\otimes\alpha\gamma+\alpha\gamma\otimes\eta-\eta\alpha\otimes\eta\gamma+\gamma\otimes\alpha),\end{aligned} \tag{3.34}$$

and

$$\Delta(b^ra^sd^tc^u) = \Lambda^r_{\tilde{k},\tilde{l},\tilde{m};k',l',m',j'}\begin{pmatrix}s\\ s'\end{pmatrix}\begin{pmatrix}u\\ u'\end{pmatrix}\ b^{\tilde{k}}a^{\tilde{l}+s'}d^{\tilde{m}+t+u'}c^{u-u'}\otimes b^{k'}a^{l'+s-s'}d^{m'+t+s'}c^{j'+u'}, \tag{3.35}$$

with summation over repeated indices and taking $i+1, j+1$ mod 2 in (3.33). The coefficients $\Gamma^{k,i,j}_{v,w;v'w'}$, $\Delta^{k,i,j}_{v,w;v'w'}$ and $\Lambda^r_{\tilde{k},\tilde{l},\tilde{m};k',l',m'j'}$ satisfy certain useful recurrence relations that allow us to reduce efficaciously the calculations (see reference [15]).

Combining (3.33) with (3.34) to construct $\Delta\mathcal{F}$, and interpreting both this result and (3.35) according to (3.32), we get

$$
\begin{aligned}
([V,W],\mathcal{F}\mathcal{B}) \;=\; & \Big\{\delta_{m0}\delta_{n0}\,\Big[\Gamma^{k,00}_{vw;0w'}\,(V,\beta^{v}\eta^{w+l}) + \Delta^{k,11}_{vw;0w'}(V,\beta^{v}\eta^{w+l})\Big] \\
+ \;& \delta_{m0}\delta_{n1}\,\Big[\Gamma^{k,00}_{vw;0w'}\,(V,\beta^{v}\eta^{w+l}\gamma) + \frac{1}{2}\Delta^{k,11}_{vw;0w'}(V,\beta^{v}\eta^{w+l}(1-\eta^{2})\alpha)\Big] \\
+ \;& \delta_{m1}\delta_{n0}\,\Big[\Gamma^{k,00}_{vw;0w'}\,(V,\beta^{v}\eta^{w+l}\alpha) + \frac{1}{2}\Delta^{k,11}_{vw;0w'}(V,\beta^{v}\eta^{w+l}(1-\eta^{2})\gamma)\Big] \\
+ \;& \delta_{m1}\delta_{n1}\,\Big[\Gamma^{k,00}_{vw;0w'}\,(V,\beta^{v}\eta^{w+l}\alpha\gamma) + \frac{1}{4}\Delta^{k,11}_{vw;0w'}(V,\beta^{v}\eta^{w+l}(1-\eta^{2})^{2})\Big]\Big\} \\
\times \;& \Lambda^{r}_{0,0,\tilde{m};k',l',m',j'}\,(W,b^{k'}a^{l'+s}d^{m'+t}c^{j'+u}) \\
- \;& \Lambda^{r}_{\tilde{k},\tilde{l},\tilde{m};0,l',m',0}\binom{s}{l'+s}\,(W,b^{\tilde{k}}a^{\tilde{l}+l'+s}d^{\tilde{m}+t}c^{u}) \\
\times \;& \Big\{\delta_{m0}\delta_{n0}\Big[\Gamma^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}) + \Delta^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l})\Big] \\
+ \;& \delta_{m0}\delta_{n1}\Big[\Gamma^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}\gamma) + \frac{1}{2}\Delta^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}(1-\eta^{2})\alpha)\Big] \\
+ \;& \delta_{m1}\delta_{n0}\Big[\Gamma^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}\alpha) + \frac{1}{2}\Delta^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}(1-\eta^{2})\gamma)\Big] \\
+ \;& \delta_{m1}\delta_{n1}\Big[\Gamma^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}\alpha\gamma) + \frac{1}{4}\Delta^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}(1-\eta^{2})^{2})\Big]\Big\}
\end{aligned}
\tag{3.36}
$$

where summation over w and w' and over repeated indices is supposed. We notice that this last expression can be yet simplified using the pairings (2.9–2.12), we get

$$
\begin{aligned}
([V,W],\mathcal{F}\mathcal{B}) \;=\; & \Big\{\delta_{m0}\delta_{n0}\,\Big[\Gamma^{k,00}_{vw;0w'}\,(V,\beta^{v}\eta^{w+l}) + \Delta^{k,11}_{vw;0w'}(V,\beta^{v}\eta^{w+l})\Big] \\
+ \;& \delta_{m0}\delta_{n1}\,\Gamma^{k,00}_{vw;0w'}\,(V,\beta^{v}\eta^{w+l}\gamma) + \delta_{m1}\delta_{n0}\,\Gamma^{k,00}_{vw;0w'}\,(V,\beta^{v}\eta^{w+l}\alpha)\Big\} \\
\times \;& \Lambda^{r}_{0,0,\tilde{m};k',l',m',j'}\,(W,b^{k'}a^{l'+s}d^{m'+t}c^{j'+u}) \\
- \;& \Lambda^{r}_{\tilde{k},\tilde{l},\tilde{m};0,l',m',0}\binom{s}{l'+s}\,(W,b^{\tilde{k}}a^{\tilde{l}+l'+s}d^{\tilde{m}+t}c^{u}) \\
\times \;& \Big\{\delta_{m0}\delta_{n0}\Big[\Gamma^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}) + \Delta^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l})\Big] \\
+ \;& \delta_{m0}\delta_{n1}\,\Gamma^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}\gamma) + \delta_{m1}\delta_{n0}\Gamma^{k,00}_{0w;v'w'}\,(V,\beta^{v'}\eta^{w'+l}\alpha)\Big\},
\end{aligned}
\tag{3.37}
$$

where summation over w and w' and over repeated indices is supposed.

For instance, if $V = A$, using the pairing (2.9), the last expression reduce to

$$
\begin{aligned}
([A,W],\mathcal{F}\mathcal{B}) \;=\; & \delta_{m1}\delta_{n0}\sum_{w,w'}\Gamma^{k,00}_{0w;0w'}\,\Big[\Lambda^{r}_{0,0,\tilde{m};k',l',m',j'}(W,b^{k'}a^{l'+s}d^{m'+t}c^{j'+u}) \\
- \;& \Lambda^{r}_{\tilde{k},\tilde{l},\tilde{m};0,l',m',0}\binom{s}{l'+s}\,(W,b^{\tilde{k}}a^{\tilde{l}+l'+s}d^{\tilde{m}+t}c^{u})\Big].
\end{aligned}
\tag{3.38}
$$

Then, using (2.13), we get

$$
\begin{aligned}
([A,\tilde{A}],\mathcal{FB}) &= \delta_{m1}\delta_{n0}\delta_{u0}\sum_{w,w'}\Gamma^{k,00}_{0w;0w'} \\
&\times \sum_{\tilde{m},m',\tilde{l},l'}\Big[\Lambda^r_{0,0,\tilde{m};0,l',m',0}\delta_{(l'+s),1} - \Lambda^r_{0,\tilde{l},\tilde{m};0,l',m',0}\binom{s}{l'+s}\delta_{(\tilde{l}+l'+s),1}\Big] \\
&= \delta_{m1}\delta_{n0}\delta_{u0}\sum_{w,w'}\Gamma^{k,00}_{0w;0w'}\sum_{\tilde{m},m'}\Big[\Lambda^r_{0,0,\tilde{m};0,1,m',0} - \Lambda^r_{0,1,\tilde{m};0,0,m',0}\Big] = 0, \quad (3.39)
\end{aligned}
$$

where the last step follow from the fact [15]

$$
\sum_{\tilde{m},m'}\Lambda^r_{0,0,\tilde{m};0,1,m',0} = \sum_{\tilde{m},m'}\Lambda^r_{0,1,\tilde{m};0,0,m',0} = 0. \quad (3.40)
$$

Moreover, from (3.38) and (2.14-2.15), we get

$$
([A,\tilde{B}],\mathcal{FB}) = \delta_{m1}\delta_{n0}\delta_{s0}\delta_{u0}\sum_{w,w'}\Gamma^{k,00}_{0w;0w'}\sum_{\tilde{m},m'}\Big[\Lambda^r_{0,0,\tilde{m};1,0,m',0} - \Lambda^r_{1,0,\tilde{m};0,0,m',0}\Big] = 0, \quad (3.41)
$$

since

$$
\sum_{\tilde{m},m'}\Lambda^r_{0,0,\tilde{m};1,0,m',0} = \sum_{\tilde{m},m'}\Lambda^r_{1,0,\tilde{m};0,0,m',0} = \delta_{r1}, \quad (3.42)
$$

and

$$
\begin{aligned}
([A,\tilde{C}],\mathcal{FB}) &= \delta_{m1}\delta_{n0}\delta_{s0}\sum_{w,w'}\Gamma^{k,00}_{0w;0w'}\sum_{\tilde{m},m'}\Big[\Lambda^r_{0,0,\tilde{m};0,0,0,m',j'}\delta_{(j'+u),1} - \Lambda^r_{0,0,\tilde{m};0,0,m',0}\delta_{u1}\Big] \\
&= \delta_{m1}\delta_{n0}\delta_{s0}\delta_{u0}\sum_{w,w'}\Gamma^{k,00}_{0w;0w'}\sum_{\tilde{m},m'}\Lambda^r_{0,0,\tilde{m};0,0,m',1} = 0, \quad (3.43)
\end{aligned}
$$

since

$$
\sum_{\tilde{m},m'}\Lambda^r_{0,0,\tilde{m};0,0,m',1} = 0. \quad (3.44)
$$

Finally, from (3.38) and (2.16), we also get

$$
([A,\tilde{D}],\mathcal{FB}) = \delta_{m1}\delta_{n0}\delta_{s0}\delta_{u0}\sum_{w,w'}\Gamma^{k,00}_{0w;0w'}\sum_{\tilde{m},m'}\Big[\tilde{m}\ \Lambda^r_{0,0,\tilde{m};0,0,m',0} - m'\ \Lambda^r_{0,0,\tilde{m};0,0,m',0}\Big] = 0, \quad (3.45)
$$

where now the last step follow from the fact

$$
\sum_{\tilde{m},m'}\tilde{m}\ \Lambda^r_{0,0,\tilde{m};0,0,m',0} = \sum_{\tilde{m},m'}m'\ \Lambda^r_{0,0,\tilde{m};0,0,m',0} = 0. \quad (3.46)
$$

In the same way, we can show that $([C,\tilde{W}],\mathcal{FB}) = 0,\ \forall\ W \in \{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\}$ and, without greater difficulties, that $([V,\tilde{W}],\mathcal{FB}) = 0,\ \forall\ V \in \{B,D\}$ and $\forall W\ \in \{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\}$.

From the above results, we observe that the direct sum structure of the initial non deformed quantum Lie superalgebra is preserved in the deformation process. Thus, using directly the results of reference [15], we get the non zero super-commutation relations

$$
\begin{aligned}
\{A, C\} &= \frac{\sinh(2B\sqrt{xz})}{2\sqrt{xz}}, \\
A^2 &= \frac{1-\cosh(2B\sqrt{xz})}{4z}, \qquad C^2 = \frac{1-\cosh(2B\sqrt{xz})}{4x}, \\
[H, A] &= -\frac{1}{2}A(1+\cosh(2B\sqrt{xz})) - x\frac{\sinh(2B\sqrt{xz})}{2\sqrt{xz}}C, \\
[H, C] &= \frac{1}{2}C(1+\cosh(2B\sqrt{xz})) + z\frac{\sinh(2B\sqrt{xz})}{2\sqrt{xz}}A,
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
[\tilde{A}, \tilde{C}] &= \frac{e^{(p+q)\tilde{B}}-1}{p+q}, \\
[\tilde{H}, \tilde{A}] &= -\tilde{A} + \rho\frac{e^{p\tilde{B}}}{p+q}\left(\frac{e^{q\tilde{B}}-1}{q} + \frac{e^{-p\tilde{B}}-1}{p}\right), \\
[\tilde{H}, \tilde{C}] &= \tilde{C} + \tau\frac{e^{q\tilde{B}}}{p+q}\left(\frac{e^{p\tilde{B}}-1}{p} + \frac{e^{-q\tilde{B}}-1}{q}\right).
\end{aligned}
\tag{3.48}
$$

4 Novel Fock space representation of the type I-II deformed quantum superalgebra

The deformed quantum superalgebra (3.47-3.48) can be represented in the Fock superspace spanned by the set of states $\{|n;j\rangle\}_{n=0}^{\infty}, j = 0, 1$, in terms of the usual fermionic $(b, b^\dagger)$ and bosonic $(a, a^\dagger)$ annihilation and creation operators associated to the supersymmetric and standard harmonic oscillator systems, respectively. Let us recall that the action of these operators on the Fock superspace is given by

$$
b\ |n;0\rangle = 0, \qquad b\ |n,1\rangle = |n,0\rangle, \qquad b^\dagger\ |n,1\rangle = 0, \qquad b^\dagger\ |n;0\rangle = |n;1\rangle \tag{4.49}
$$

and

$$
a\ |n,j\rangle = \sqrt{n}\ |n-1,j\rangle, \qquad a^\dagger\ |n,j\rangle = \sqrt{n+1}\ |n+1,j\rangle. \tag{4.50}
$$

Thus the fermionic oscillator sub-superalgebra can be realized in terms of the usual fermionic operators $b, b^\dagger$ and the identity I, by taking, for instance (see Appendix B),

$$
A = b + \frac{1-\cosh(2\sqrt{xz})}{4z}\ b^\dagger, \qquad B = I, \qquad H = \cosh(\sqrt{xz})\ b^\dagger b, \tag{4.51}
$$

$$
C = \frac{\cosh(\sqrt{xz})+1}{2\sqrt{xz}}\sinh(\sqrt{xz})\ b^\dagger - \sqrt{\frac{z}{x}}\ \frac{\sinh(\sqrt{xz})}{\cosh(\sqrt{xz})+1}\ b. \tag{4.52}
$$

When $x \neq 0$, another choice for A and C (see Appendix B) is

$$A = \frac{\sinh(\sqrt{xz})}{\sqrt{2z}}(b - b^\dagger), \qquad C = \frac{\cosh(\sqrt{xz}) + 1}{\sqrt{2x}}\; b^\dagger - \frac{\cosh(\sqrt{xz}) - 1}{\sqrt{2x}}\; b. \tag{4.53}$$

Let us insist on the fact that we have chosen two realizations for which A and C are linear combinations of b and $b^\dagger$. In the first cas (4.51– 4.52), A is a deformation of b and C of $b^\dagger$, whereas in the second one (4.53), when z goes to zero, $A \mapsto \sqrt{\frac{x}{2}}\,(b - b^\dagger)$ and $C \mapsto \sqrt{\frac{2}{x}}\, b^\dagger$.

On the other hand, a natural realization of the bosonic oscillator subalgebra in terms of the operators $a, a^\dagger$ and I, is given by

$$\tilde{A} \;=\; \sqrt{\omega}\; a, \qquad \tilde{B} = I, \qquad \tilde{C} = \sqrt{\omega}\; a^\dagger, \qquad \tilde{H} = a^\dagger a + \tilde{\tau}\; a - \tilde{\rho}\; a^\dagger, \tag{4.54}$$

where $w = \frac{e^{(p+q)}-1}{p+q}$ and

$$\tilde{\tau} = \frac{\tau}{\sqrt{\omega}}\frac{e^q}{p+q}\left(\frac{e^p - 1}{p} + \frac{e^{-q} - 1}{q}\right), \qquad \tilde{\rho} = \frac{\rho}{\sqrt{\omega}}\frac{e^p}{p+q}\left(\frac{e^q - 1}{q} + \frac{e^{-p} - 1}{p}\right). \tag{4.55}$$

Thus, the commutation relations (3.48), can be obtained from

$$[\tilde{A}, \tilde{C}] = \omega I, \qquad [\tilde{H}_0, \tilde{A}] = -\tilde{A}, \qquad [\tilde{H}_0, \tilde{C}] = \tilde{C}, \tag{4.56}$$

where

$$\tilde{H}_0 = a^\dagger a + \tilde{\tau}\; \tilde{\rho}\; I, \tag{4.57}$$

by performing the transformation

$$T = \exp\left(\tilde{\tau}\; a\right)\; \exp\left(\tilde{\rho}\; a^\dagger\right). \tag{4.58}$$

We notice that T is not an unitary operator except for the special case when $\tilde{\rho} = -\tilde{\tau}$.

Let us remark that these results represent new realizations of the deformed fermionic type I (see Appendix B) and deformed type II quantum Lie algebras that have not been obtained before in the literature.

It is clear that, in the limit when the deformation parameters goes to zero, in all the cases, we regain essentially the standard representation of the super-oscillator Lie superalgebra $ho(2|2, \mathbb{R}) \oplus ho(4, \mathbb{R})$:

$$A \;=\; b, \qquad B = I, \qquad C = b^\dagger, \qquad H = b^\dagger b, \tag{4.59}$$

$$\tilde{A} \;=\; a, \qquad \tilde{B} = I, \qquad \tilde{C} = a^\dagger, \qquad \tilde{H} = a^\dagger a. \tag{4.60}$$

5 Deformed coherent states

To be able to construct super-coherent states associated to the deformed superalgebra (3.47–3.48), we construct a suitable deformed annihilation operator $\mathbb{A}_0$. A possible choice is

$$\mathbb{A}_0 = \tilde{A} + A, \tag{5.1}$$

with A and $\tilde{A}$ as given by equations (4.51) and (4.54) or (4.53) and (4.54), respectively. The first case corresponds to the deformed super-annihilator associated to the supersymmetric harmonic oscillator, introduced by Aragone and Zypmann[16]. The second one corresponds to a deformed annihilator associated to a generalized Hamiltonian isospectral to the harmonic oscillator one but two times degenerated [17].

5.1 Deformed super-coherent states

Using (4.51) and (4.54), the annihilator $\mathbb{A}_0$ becomes

$$\mathbb{A}_0 = \sqrt{\omega}\; a + b + \frac{1 - \cosh(2\sqrt{xz})}{4z}\; b^\dagger. \tag{5.2}$$

Thus, the deformed coherent states $|Z; x, z, \omega\rangle$, i.e., the eigenstates of $\mathbb{A}_0$ associated to the eigenvalue $Z \in \mathbb{C}$, verify the eigenvalue equation

$$\left[\sqrt{\omega}\; a + b + \frac{1 - \cosh(2\sqrt{xz})}{4z}\; b^\dagger\right]\; |Z; x, z, \omega\rangle = Z\; |Z; x, z, \omega\rangle. \tag{5.3}$$

The two independent solutions of this eigenvalue equation has been obtained in reference [17]. They are given by the normalized eigenstates

$$|Z; x, z, \omega; \mp\rangle = D\left(\frac{Z}{\sqrt{\omega}} + \frac{i}{\sqrt{2z\omega}}\sinh(\sqrt{xz})\right) \frac{\left[|0;0\rangle \mp \frac{i}{\sqrt{2z}}\sinh(\sqrt{xz})|0;1\rangle\right]}{\sqrt{1 + \frac{\sinh^2(\sqrt{xz})}{2z}}}, \tag{5.4}$$

where $D(\alpha)$ is the usual displacement unitary operator [18]

$$D(\alpha) = \exp\left(\alpha a^\dagger - \bar{\alpha} a\right). \tag{5.5}$$

In absence of deformation, i.e., in the limit when z and x tend to zero, the deformed coherent states (5.4) becomes the usual coherent states associated to the standard harmonic oscillator

$$|Z; 0, 0, \omega; \mp\rangle = D\left(\frac{Z}{\sqrt{\omega}}\right) |0; 0\rangle. \tag{5.6}$$

In this case, there is another independent solution to the eigenvalue equation (5.3). It is obtained by solving directly $[\sqrt{\omega}a + b]|Z\rangle = Z|Z\rangle$ and is exactly the super-coherent states, given by Aragone and Zypmann [16], associated to the super-symmetric harmonic oscillator:

$$|Z;0,0,\omega\rangle = D\left(\frac{Z}{\sqrt{\omega}}\right)\frac{[|1;0\rangle - |0;1\rangle]}{\sqrt{2}}. \tag{5.7}$$

5.2 Isospectral harmonic oscillator systems

When $x \neq 0$, using (4.53) and (4.54), the annihilator $\mathbb{A}_0$ becomes

$$\mathbb{A}_0 = \sqrt{\omega}\; a + \frac{\sinh(\sqrt{xz})}{\sqrt{2z}}(b - b^\dagger) \tag{5.8}$$

and verifies the canonical commutation relation

$$[\mathbb{A}_0, \mathbb{A}_0^\dagger] = wI. \tag{5.9}$$

In this case, the deformed coherent states $|Z;x,z,\omega\rangle$, satisfy the eigenvalue equation

$$\left[\sqrt{\omega}\; a + \frac{\sinh(\sqrt{xz})}{\sqrt{2z}}(b - b^\dagger)\right]\; |Z;x,z,\omega\rangle = Z\; |Z;x,z,\omega\rangle. \tag{5.10}$$

Two independent and orthogonal solutions to this eigenvalue equation are (see [17])

$$|Z;x,z,\omega;\mp\rangle = D\left(\frac{Z}{\sqrt{\omega}} + i\frac{\sinh(\sqrt{xz})}{\sqrt{2z\omega}}\right)\frac{[|0;0\rangle \mp i|0;1\rangle]}{\sqrt{2}}. \tag{5.11}$$

Let us define the Hermitian Hamiltonian

$$\mathbb{H}_0 = \mathbb{A}_0^\dagger\mathbb{A}_0 = \omega a^\dagger a + \frac{\sinh^2(\sqrt{xz})}{2z}I + \sqrt{\frac{\omega}{2z}}\sinh\sqrt{xz}(a^\dagger b + ab^\dagger - a^\dagger b^\dagger - ab). \tag{5.12}$$

Then, from (5.9), we get

$$[\mathbb{H}_0, \mathbb{A}_0] = -w\mathbb{A}_0, \qquad [\mathbb{H}_0, \mathbb{A}_0^\dagger] = w\mathbb{A}_0^\dagger, \tag{5.13}$$

i.e., $\mathbb{A}_0$ and $\mathbb{A}_0^\dagger$ are the annihilation and creation operators associated to $\mathbb{H}_0$. By construction, the eigenstates of $\mathbb{H}_0$, corresponding to the energy eigenvalue $E_0 = 0$, are given by the deformed coherent states (5.11), when $Z = 0$, i.e.,

$$|0;x,z,\omega;\mp\rangle = D\left(i\frac{\sinh(\sqrt{xz})}{\sqrt{2z\omega}}\right)\frac{[|0;0\rangle \mp i|0;1\rangle]}{\sqrt{2}}. \tag{5.14}$$

Then, from (5.13), the eigenstates of $\mathbb{H}_0$ corresponding to the energy eigenvalues $E_n = nw, n = 0, 1, 2, \ldots$, are given by

$$|E_n; x, z, \omega; \mp\rangle = \frac{\left(\mathbb{A}_0^\dagger\right)^n}{\sqrt{n!}} |0; x, z, \omega; \mp\rangle. \tag{5.15}$$

Thus, $\mathbb{H}_0$ is isospectral to the standard harmonic oscillator Hamitonian but two times degenerated. We notice that this Hamiltonian is an element of the $osp(2/2) \Supset sh(2/2)$ superalgebra.

By analogy with the standard harmonic oscillator system, the coherent states associated to $\mathbb{H}_0$ can be obtained by acting with the unitary displacement operator

$$\mathbb{D}(\alpha) = \exp\left(\alpha\mathbb{A}_0^\dagger - \bar{\alpha}\mathbb{A}_0\right) \tag{5.16}$$

on the zero energy eigenstate, that is

$$|Z; \widetilde{x, z, \omega}; \mp\rangle = \mathbb{D}\left(\frac{Z}{\sqrt{w}}\right) |0; x, z, \omega; \mp\rangle. \tag{5.17}$$

These last states can also be obtained from (5.11) by acting with the unitary operator

$$\mathbb{U} = \exp\left(\frac{\sqrt{2}}{\omega}\,\mathrm{Re}\,Z\,\frac{\sinh\sqrt{xz}}{\sqrt{z}}(b - b^\dagger)\right). \tag{5.18}$$

Let us now consider the transformation (4.58) and define the Hamiltonian

$$\mathbb{H} = \tilde{\mathbb{A}}\mathbb{A} = T\mathbb{H}_0T^{-1}, \tag{5.19}$$

with

$$\mathbb{A} = T\mathbb{A}_0T^{-1} \quad \text{and} \quad \tilde{\mathbb{A}} = T\mathbb{A}_0^\dagger T^{-1}. \tag{5.20}$$

This Hamiltonian verifies the η–pseudo-Hermiticity property [19]

$$\mathbb{H}^\dagger = \eta\mathbb{H}\eta^{-1}, \tag{5.21}$$

where η is the Hermitian operator

$$\eta = (T^{-1})^\dagger T^{-1} = e^{-\tilde{\tau}a^\dagger}e^{-\tilde{\rho}a}e^{-\tilde{\rho}a^\dagger}e^{-\tilde{\tau}a}. \tag{5.22}$$

By applying the transformation T to the commutation relations (5.9) and (5.13), we get

$$[\mathbb{A}, \tilde{\mathbb{A}}] = wI, \qquad [\mathbb{H}, \mathbb{A}] = -w\mathbb{A}, \qquad [\mathbb{H}, \tilde{\mathbb{A}}] = w\tilde{\mathbb{A}}, \tag{5.23}$$

i.e., $\mathbb{A}$, $\tilde{\mathbb{A}}$, I and $\mathbb{H}$ verify the commutation relation of the oscillator Lie algebra $ho(4,\mathbb{R})$. Thus, $\mathbb{A}$ and $\tilde{\mathbb{A}}$ are the annihilation and creation operators associated to $\mathbb{H}$. This implies that the Hamiltonian $\mathbb{H}$ has the same energy spectrum than $\mathbb{H}_0$ and their eigenstates are given by $T|E_n;x,z,\omega;\mp\rangle$. The coherent states associated to $\mathbb{H}$ are the eigenstates of $\mathbb{A}$, i.e., $T|Z;x,z,\omega;\mp\rangle$. These coherent states include both the x,z and the ρ,τ deformation parameters and can also be considered as a class of coherent states associated to the type I-II fermionic-bosonic quantum Lie algebra (3.47-3.48) in the chosen realization.

6 Conclusion

In this work, using the R–matrix method, we have computed the two oscillator fermionic-bosonic type I-II quantum bialgebras and the corresponding dual quantum Lie superalgebras. We have shown that, in general, though the quantum bialgebras associated with a two independent oscillator group are not the structure of a direct sum of bialgebras associated to the single oscillator group the dual quantum Lie algebras have the structure of a direct sum of quantum Lie algebras associated to the single oscillator group. Then we have given some examples of the type I-II fermionic-bosonic quantum Lie algebras and several realizations of them in terms of the usual fermionic and bosonic creation and annihilation operators. Based on this realization, we have also found a class of coherent states associated with the two oscillator quantum group. The other types of fermionic-bosonic quantum algebras, that we have mentioned in section 3, can be treated in the same way.

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A Determination of the possible R matrices and the bialgebras structures

From (2.21), setting to zero every linear relation between the generators, we get the block matrices $R^{i,j}$:

$$R^{11} = \begin{pmatrix} r^{11}_{11} & 0 & r^{11}_{13} & 0 & 0 \\ 0 & r^{11}_{11} & 0 & 0 & 0 \\ 0 & 0 & r^{11}_{11} & 0 & 0 \\ 0 & 0 & 0 & r^{11}_{11} & 0 \\ 0 & 0 & r^{11}_{53} & 0 & r^{11}_{11} \end{pmatrix}, \qquad R^{12} = \begin{pmatrix} 0 & r^{12}_{12} & r^{12}_{13} & 0 & 0 \\ 0 & r^{12}_{22} & r^{12}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^{12}_{43} & r^{12}_{44} & 0 \\ 0 & 0 & r^{12}_{53} & r^{12}_{54} & 0 \end{pmatrix}, \tag{A.1}$$

$$R^{13} = \begin{pmatrix} r^{13}_{11} & r^{13}_{12} & r^{13}_{13} & 0 & 0 \\ 0 & r^{13}_{22} & r^{13}_{23} & 0 & 0 \\ 0 & 0 & r^{13}_{33} & 0 & 0 \\ 0 & 0 & r^{13}_{43} & r^{13}_{44} & 0 \\ r^{13}_{51} & 0 & r^{13}_{53} & r^{13}_{54} & r^{13}_{55} \end{pmatrix}, \tag{A.2}$$

$$R^{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^{14}_{43} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R^{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^{15}_{53} & 0 & 0 \end{pmatrix}, \tag{A.3}$$

$$r^{14}_{43} = r^{15}_{53}, \tag{A.4}$$

$$R^{22} = \begin{pmatrix} r^{22}_{11} & r^{22}_{12} & r^{22}_{13} & 0 & 0 \\ 0 & r^{22}_{22} & r^{22}_{23} & 0 & 0 \\ 0 & 0 & r^{22}_{11} & 0 & 0 \\ 0 & 0 & r^{22}_{43} & r^{22}_{44} & 0 \\ 0 & 0 & r^{22}_{53} & r^{22}_{54} & r^{22}_{11} \end{pmatrix} \qquad R^{23} = \begin{pmatrix} 0 & r^{23}_{12} & r^{23}_{13} & 0 & 0 \\ 0 & r^{23}_{22} & r^{23}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^{23}_{43} & r^{23}_{44} & 0 \\ 0 & r^{23}_{52} & r^{23}_{53} & r^{23}_{54} & 0 \end{pmatrix}, \tag{A.5}$$

$$R^{21} = R^{24} = R^{25} = 0, \qquad r^{23}_{52} = r^{13}_{51}, \qquad r^{22}_{11} = r^{11}_{11}, \tag{A.6}$$

$$R^{33} = \begin{pmatrix} r^{33}_{11} & 0 & r^{33}_{13} & 0 & 0 \\ 0 & r^{33}_{11} & 0 & 0 & 0 \\ 0 & 0 & r^{33}_{11} & 0 & 0 \\ 0 & 0 & 0 & r^{33}_{11} & 0 \\ 0 & 0 & r^{33}_{53} & 0 & r^{33}_{11} \end{pmatrix}, \tag{A.7}$$

$$R^{31} = R^{32} = R^{34} = R^{35} = 0, \qquad r^{33}_{11} = r^{11}_{11}, \tag{A.8}$$

$$R^{43}=\begin{pmatrix}0 & r^{43}_{12} & r^{43}_{13} & r^{43}_{14} & 0\\ 0 & r^{43}_{22} & r^{43}_{23} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & r^{43}_{43} & r^{43}_{44} & 0\\ 0 & 0 & r^{43}_{53} & r^{43}_{54} & 0\end{pmatrix}, \qquad R^{44}=\begin{pmatrix}r^{44}_{11} & r^{44}_{12} & r^{44}_{13} & 0 & 0\\ 0 & r^{44}_{22} & r^{44}_{23} & 0 & 0\\ 0 & 0 & r^{44}_{11} & 0 & 0\\ 0 & 0 & r^{44}_{43} & r^{44}_{44} & 0\\ 0 & 0 & r^{44}_{53} & r^{44}_{54} & r^{44}_{11}\end{pmatrix}, \tag{A.9}$$

$$R^{41}=R^{42}=R^{45}=0, \qquad r^{44}_{11}=r^{11}_{11} \qquad r^{43}_{14}=r^{53}_{15}, \tag{A.10}$$

$$R^{51}=\begin{pmatrix}0 & 0 & r^{51}_{13} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} \qquad R^{52}=\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & r^{52}_{23} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}, \tag{A.11}$$

$$R^{53}=\begin{pmatrix}r^{53}_{11} & r^{53}_{12} & r^{53}_{13} & 0 & r^{53}_{15}\\ 0 & r^{53}_{22} & r^{53}_{23} & 0 & 0\\ 0 & 0 & r^{53}_{33} & 0 & 0\\ 0 & 0 & r^{53}_{43} & r^{53}_{44} & 0\\ 0 & 0 & r^{53}_{53} & r^{53}_{54} & r^{53}_{55}\end{pmatrix}, \tag{A.12}$$

$$R^{54}=\begin{pmatrix}0 & r^{54}_{12} & r^{54}_{13} & 0 & 0\\ 0 & r^{54}_{22} & r^{54}_{23} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & r^{54}_{43} & r^{54}_{44} & 0\\ 0 & 0 & r^{54}_{53} & r^{54}_{54} & 0\end{pmatrix}, \qquad R^{55}=\begin{pmatrix}r^{55}_{11} & 0 & r^{55}_{13} & 0 & 0\\ 0 & r^{55}_{11} & 0 & 0 & 0\\ 0 & 0 & r^{55}_{11} & 0 & 0\\ 0 & 0 & 0 & r^{55}_{11} & 0\\ 0 & 0 & r^{55}_{53} & 0 & r^{55}_{11}\end{pmatrix}, \tag{A.13}$$

$$r^{55}_{11}=r^{11}_{11}, \qquad r^{51}_{13}=r^{52}_{23}. \tag{A.14}$$

A.1 Deforming the femionic-bosonic bialgebra

We are interested to deform the Lie superalgebra $ho(2/2,\mathbb{R})\oplus ho(4,\mathbb{R})$. Choosing as reference point the non deformed bialgebra (2.28-2.30), we get the following set of quadratic relations:

$$\begin{aligned}
[\alpha,\beta] &= r^{22}_{23}\alpha^2 - r^{12}_{12}\eta\gamma - (r^{11}_{13}-r^{12}_{23}-r^{22}_{13})\alpha + r^{12}_{13}(1-\eta), && \text{(A.15)}\\
[\beta,\gamma] &= -r^{12}_{22}\gamma^2 + r^{23}_{23}\eta\alpha - (r^{12}_{23}+r^{13}_{22}-r^{13}_{33})\gamma - r^{13}_{23}(1-\eta), && \text{(A.16)}\\
[\alpha,\gamma] &= r^{22}_{23}\eta\alpha - r^{12}_{22}\eta\gamma, && \text{(A.17)}\\
[\eta,\beta] &= r^{22}_{23}\alpha\eta - r^{12}_{22}\eta\gamma, && \text{(A.18)}
\end{aligned}$$

$$\{\alpha,\eta\} = r^{12}_{22}(\eta-\eta^2), \tag{A.19}$$

$$\{\gamma,\eta\} = r^{22}_{23}(\eta-\eta^2), \tag{A.20}$$

$$\alpha^2 = \frac{1}{2}r^{12}_{12}(1-\eta^2)+r^{12}_{22}\alpha, \tag{A.21}$$

$$\gamma^2 = \frac{1}{2}r^{23}_{23}(1-\eta^2)+r^{22}_{23}\gamma, \tag{A.22}$$

$$[a,b] = r^{44}_{43}a^2-(r^{55}_{53}-r^{54}_{43}-r^{44}_{53})a+r^{54}_{53}(1-d), \tag{A.23}$$

$$[b,c] = -r^{54}_{44}c^2-(r^{54}_{43}+r^{53}_{44}-r^{53}_{33})c-r^{53}_{43}(1-d), \tag{A.24}$$

$$[a,c] = r^{44}_{43}da-r^{54}_{44}dc, \tag{A.25}$$

$$[d,b] = r^{44}_{43}ad+r^{54}_{44}dc, \tag{A.26}$$

$$[a,d] = r^{54}_{44}(d-d^2), \tag{A.27}$$

$$[c,d] = r^{44}_{43}(d-d^2), \tag{A.28}$$

$$[\eta,a] = r^{22}_{54}\eta(1-d), \tag{A.29}$$

$$[\eta,b] = r^{22}_{43}a\eta-r^{22}_{54}\eta c, \tag{A.30}$$

$$[\eta,c] = r^{22}_{43}(d-1)\eta, \tag{A.31}$$

$$[\eta,d] = 0, \tag{A.32}$$

$$[\alpha,a] = r^{22}_{54}\alpha+r^{12}_{44}a+r^{12}_{54}(1-\eta d), \tag{A.33}$$

$$[\alpha,c] = r^{12}_{43}(d-\eta)+r^{22}_{43}d\alpha-r^{12}_{44}\eta c, \tag{A.34}$$

$$[\alpha,b] = (r^{22}_{53}-r^{11}_{53})\alpha+r^{12}_{43}a+r^{22}_{43}a\alpha-r^{12}_{54}\eta c+r^{12}_{53}(1-\eta), \tag{A.35}$$

$$[\alpha,d] = r^{44}_{12}d(\eta-1), \tag{A.36}$$

$$[c,\gamma] = r^{22}_{43}\gamma+r^{23}_{44}c+r^{23}_{43}(1-d\eta), \tag{A.37}$$

$$[\gamma,a] = r^{23}_{44}a\eta-r^{22}_{54}\gamma\eta, \tag{A.38}$$

$$[\gamma,b] = r^{23}_{53}(\eta-1)-(r^{22}_{53}+r^{23}_{52}-r^{33}_{53})\gamma-r^{23}_{54}c-r^{22}_{54}\gamma c+r^{23}_{43}a\eta, \tag{A.39}$$

$$[\gamma,d] = r^{44}_{23}(1-\eta)d, \tag{A.40}$$

$$[\beta,a] = r^{23}_{54}\alpha+(r^{13}_{44}-r^{13}_{55})a+r^{13}_{54}(1-d)+r^{23}_{44}a\alpha-r^{12}_{54}\gamma d, \tag{A.41}$$

$$[\beta,c] = -r^{12}_{43}\gamma+(r^{13}_{33}-r^{13}_{44}-r^{14}_{43})c+r^{13}_{43}(d-1)-r^{12}_{44}\gamma c+r^{23}_{43}d\alpha, \tag{A.42}$$

$$\begin{aligned}[\beta,b] &= r^{23}_{53}\alpha-(r^{13}_{51}+r^{11}_{53}-r^{33}_{53})\beta-r^{12}_{53}\gamma+r^{13}_{43}a\\ &+ (r^{13}_{33}-r^{15}_{53}-r^{13}_{55})b-r^{13}_{54}c-r^{12}_{54}\gamma c+r^{23}_{43}a\alpha,\end{aligned} \tag{A.43}$$

$$[\beta,d] = r^{44}_{23}(1-\eta)d. \tag{A.44}$$

We have also the consistency relations

$$r_{23}^{23}r_{22}^{12} = 0, \qquad r_{12}^{12}r_{23}^{22} = 0, \tag{A.45}$$

$$r_{13}^{12} + r_{12}^{13} = 0, \qquad r_{12}^{22} = r_{22}^{12}, \qquad r_{22}^{23} = r_{23}^{22}, \tag{A.46}$$

$$r_{11}^{13} + r_{13}^{11} = r_{33}^{13} + r_{13}^{33}, \qquad r_{23}^{13} + r_{13}^{23} = 0, \tag{A.47}$$

$$2r_{23}^{22}r_{22}^{12} = (r_{11}^{13} - r_{12}^{23} - r_{22}^{13}) + (r_{13}^{11} - r_{23}^{12} - r_{13}^{22}), \tag{A.48}$$

$$r_{54}^{44} = -r_{44}^{54}, \qquad r_{44}^{43} = -r_{43}^{44}, \qquad r_{54}^{54} = r_{43}^{43} = 0, \tag{A.49}$$

$$r_{55}^{53} + r_{53}^{55} = r_{33}^{53} + r_{53}^{33}, \qquad r_{43}^{53} + r_{53}^{43} = 0, \qquad r_{53}^{54} + r_{54}^{53} = 0, \tag{A.50}$$

$$r_{55}^{53} - r_{54}^{43} - r_{44}^{53} = r_{43}^{54} + r_{53}^{44} - r_{53}^{55} = p, \qquad r_{43}^{54} + r_{44}^{53} - r_{33}^{53} = r_{53}^{33} - r_{54}^{43} - r_{53}^{44} = q, \tag{A.51}$$

$$r_{22}^{43} + r_{43}^{22} = 0, \qquad r_{22}^{54} + r_{54}^{22} = 0, \tag{A.52}$$

$$r_{23}^{44} + r_{44}^{23} = 0, \qquad r_{44}^{12} + r_{12}^{44} = 0, \tag{A.53}$$

$$r_{23}^{43} + r_{43}^{23} = 0, \qquad r_{54}^{12} + r_{12}^{54} = 0, \tag{A.54}$$

$$r_{54}^{23} + r_{23}^{54} = -r_{23}^{44}r_{54}^{22}, \qquad r_{43}^{12} + r_{12}^{43} = -r_{43}^{22}r_{12}^{44}, \tag{A.55}$$

$$r_{53}^{12} + r_{12}^{53} = -r_{54}^{12}r_{22}^{43}, \qquad r_{54}^{13} + r_{13}^{54} = -r_{54}^{12}r_{23}^{44}, \tag{A.56}$$

$$r_{43}^{13} + r_{13}^{43} = r_{44}^{12}r_{43}^{23}, \qquad r_{23}^{53} + r_{53}^{23} = -r_{43}^{23}r_{22}^{54}, \tag{A.57}$$

$$r_{44}^{13} + r_{13}^{44} - r_{55}^{13} - r_{13}^{55} = r_{44}^{12}r_{44}^{23}, \qquad r_{33}^{13} + r_{13}^{33} - r_{44}^{13} - r_{13}^{44} - r_{43}^{14} - r_{14}^{43} = -r_{44}^{12}r_{44}^{23}, \tag{A.58}$$

$$r_{53}^{22} + r_{22}^{53} - r_{53}^{11} - r_{11}^{53} = r_{43}^{22}r_{54}^{22}, \qquad r_{33}^{53} + r_{53}^{33} - r_{22}^{53} - r_{53}^{22} - r_{23}^{52} - r_{52}^{23} = -r_{43}^{22}r_{54}^{22}. \tag{A.59}$$

There are several possible choices for the remaining entries of these R matrices so that they verify the QYBE. Indeed, the different forms of fixing the coefficients in equations equations (A.45) and (A.49) determine the different types of deformation of the fermionic-bosonic bialgebra (2.28-2.30), i.e., the type I–I, I–II, I–III II–I II-II and II–III.

B Realization of the fermionic type I quantum superalgebra

In this section we give some realizations for the type I fermionic quantum superalgebra (3.47), in terms of the physical generators (4.59).

Let us consider the linear combinations

$$A = a_0 I + a_1 b + a_2 b^\dagger + a_3 b^\dagger b, \tag{B.1}$$

$$B = I, \tag{B.2}$$

$$C = c_0 I + c_1 b + c_2 b^\dagger + c_3 b^\dagger b, \tag{B.3}$$

$$H = h_0 I + h_1 b + h_2 b^\dagger + h_3 b^\dagger b, \tag{B.4}$$

where $a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3, h_0, h_1, h_2$ and h_3 are, in general, complex coefficients to determine. Inserting (B.1–B.4) into (3.47), we get the following system of algebraic equations

$$a_3 = -2a_0, \qquad c_3 = -2c_0, \tag{B.5}$$

$$a_0^2 + a_1 a_2 = \frac{1-\cosh(2\sqrt{xz})}{4z}, \tag{B.6}$$

$$c_0^2 + c_1 c_2 = \frac{1-\cosh(2\sqrt{xz})}{4x}, \tag{B.7}$$

$$2a_0 c_0 + a_1 c_2 + a_2 c_1 = \frac{\sinh(2\sqrt{xz})}{2\sqrt{xz}}, \tag{B.8}$$

$$h_1 a_2 - h_2 a_1 = -\frac{1}{2}(1+\cosh(2\sqrt{xz})a_0 - x\frac{\sinh(2\sqrt{xz})}{2\sqrt{xz}}c_0, \tag{B.9}$$

$$h_3 a_1 + 2h_1 a_0 = \frac{1}{2}(1+\cosh(2\sqrt{xz})a_1 + x\frac{\sinh(2\sqrt{xz})}{2\sqrt{xz}}c_1, \tag{B.10}$$

$$h_3 a_2 + 2h_2 a_0 = -\frac{1}{2}(1+\cosh(2\sqrt{xz})a_2 - x\frac{\sinh(2\sqrt{xz})}{2\sqrt{xz}}c_2, \tag{B.11}$$

$$h_1 c_2 - h_2 c_1 = \sqrt{\frac{z}{x}}\frac{\sinh(2\sqrt{xz})}{2}a_0 + \frac{1}{2}(1+\cosh(2\sqrt{xz})c_0, \tag{B.12}$$

$$h_3 c_1 + 2h_1 c_0 = -\sqrt{\frac{z}{x}}\frac{\sinh(2\sqrt{xz})}{2}a_1 - \frac{1}{2}(1+\cosh(2\sqrt{xz})c_1, \tag{B.13}$$

$$\tag{B.14}$$

$$h_3 c_2 + 2h_2 c_0 = \sqrt{\frac{z}{x}}\frac{\sinh(2\sqrt{xz})}{2}a_2 + \frac{1}{2}(1+\cosh(2\sqrt{xz})c_2. \tag{B.15}$$

There are several solutions to this system of equations which depend on a subset of arbitrary coefficients. For example,

$$c_1 = c_2 = 0 = h_3 = 0, \qquad a_0 = \pm i\frac{\cosh\sqrt{xz}}{\sqrt{2z}}, \qquad a_1 = \frac{1}{2za_2}, \tag{B.16}$$

$$c_0 = \mp i\frac{\sinh\sqrt{xz}}{\sqrt{2x}}, \qquad h_1 = \mp i\frac{\cosh\sqrt{xz}}{2\sqrt{2z}a_2}, \qquad h_2 = \pm i\sqrt{\frac{z}{2}}a_2, \cosh\sqrt{xz}, \tag{B.17}$$

for arbitrary h_0 and $a_2 \neq 0$. Thus, a realization of the fermionic superalgebra (3.47), is

$$
\begin{aligned}
A &= \pm i\frac{\cosh\sqrt{xz}}{\sqrt{2z}}\ I + \frac{1}{2za_2}\ b + a_2\ b^\dagger \mp 2i\frac{\cosh\sqrt{xz}}{\sqrt{2z}}\ b^\dagger b, && \text{(B.18)}\\
B &= I, && \text{(B.19)}\\
C &= \mp i\frac{\sinh\sqrt{xz}}{\sqrt{2x}}\ I \pm 2i\frac{\sinh\sqrt{xz}}{\sqrt{2x}}\ b^\dagger b, && \text{(B.20)}\\
H &= h_0\ I \mp i\frac{\cosh\sqrt{xz}}{2a_2\sqrt{2z}}\ b \pm \sqrt{\frac{z}{2}}a_2\cosh\sqrt{xz}\ b^\dagger. && \text{(B.21)}
\end{aligned}
$$

In the case when $a_0 = c_2 = 0$, we have

$$
\begin{aligned}
a_1 &= \frac{1-\cosh(2\sqrt{xz})}{4za_2}, \qquad c_0 = -i\frac{\sinh\sqrt{xz}}{\sqrt{2x}}, \qquad c_1 = \frac{\sinh\sqrt{xz}}{2a_2\sqrt{xz}}, && \text{(B.22)}\\
h_1 &= i\frac{\cosh(4\sqrt{xz})-1}{16a_2\sqrt{2z}\cosh\sqrt{xz}}, \qquad h_2 = ia_2\sqrt{\frac{z}{x}}\frac{\cosh(4\sqrt{xz})-1}{4\sqrt{2}\cosh\sqrt{xz}}, && \text{(B.23)}\\
h_3 &= -\frac{\cosh(3\sqrt{xz})+3\cosh\sqrt{xz}}{4\cosh\sqrt{xz}}, && \text{(B.24)}
\end{aligned}
$$

for arbitrary h_0 and $a_2 \neq 0$. In this case, the realization of the deformed Lie algebra is given by

$$
\begin{aligned}
A &= \frac{1-\cosh(2\sqrt{xz})}{4za_2}\ b + a_2\ b^\dagger, && \text{(B.25)}\\
B &= I, && \text{(B.26)}\\
C &= -i\frac{\sinh\sqrt{xz}}{\sqrt{2x}}\ I + \frac{\sinh\sqrt{xz}}{2a_2\sqrt{xz}}\ b + 2i\frac{\sinh\sqrt{xz}}{\sqrt{2x}}\ b^\dagger b, && \text{(B.27)}\\
H &= h_0\ I + i\frac{\cosh(4\sqrt{xz})-1}{16a_2\sqrt{2z}\cosh\sqrt{xz}}\ b + ia_2\sqrt{\frac{z}{x}}\frac{\cosh(4\sqrt{xz})-1}{4\sqrt{2}\cosh\sqrt{xz}}\ b^\dagger\\
&- \frac{\cosh(3\sqrt{xz})+3\cosh\sqrt{xz}}{4\cosh\sqrt{xz}}\ b^\dagger b. && \text{(B.28)}
\end{aligned}
$$

In the case when $c_0 = a_2 = 0$, we have

$$
\begin{aligned}
a_0 &= \pm i\frac{\sinh\sqrt{xz}}{\sqrt{2z}}, \qquad a_1 = -2\sqrt{\frac{x}{z}}\ c_1\coth\sqrt{xz}, \qquad c_2 = -\frac{\sinh^2\sqrt{xz}}{2xc_1}, && \text{(B.29)}\\
h_1 &= \mp i\sqrt{\frac{x}{2}}\ c_1\cosh\sqrt{xz}, \qquad h_2 = \mp i\frac{\sinh^2\sqrt{xz}}{2\sqrt{2x}\ c_1}\cosh\sqrt{xz}, && \text{(B.30)}\\
h_3 &= \cosh^2\sqrt{xz}, && \text{(B.31)}
\end{aligned}
$$

for arbitrary h_0 and $c_1 \neq 0$. The realizations of (3.47), are given by

$$
A = \pm i\frac{\sinh\sqrt{xz}}{\sqrt{2z}}\ I - 2\sqrt{\frac{x}{z}}c_1\coth\sqrt{xz}\ b \mp 2i\frac{\sinh\sqrt{xz}}{\sqrt{2z}}\ b^\dagger b, \qquad \text{(B.32)}
$$

$$
\begin{aligned}
B &= I, && \text{(B.33)}\\
C &= c_1\, b - \frac{\sinh^2\sqrt{xz}}{2xc_1}\, b^\dagger, && \text{(B.34)}\\
H &= h_0\, I \mp i\sqrt{\frac{x}{2}}\, c_1 \cosh\sqrt{xz}\, b \mp i\frac{\sinh^2\sqrt{xz}}{2\sqrt{2x}\, c_1}\cosh\sqrt{xz}\, b^\dagger \\
&+ \cosh^2\sqrt{xz}\, b^\dagger b. && \text{(B.35)}
\end{aligned}
$$

On the other hand if we take $a_0 = c_0 = 0$, we get

$$
\begin{aligned}
a_1 &= \frac{(\cosh\sqrt{xz}\mp 1)}{2c_2\sqrt{xz}}\sinh\sqrt{xz}, \qquad a_2 = -\sqrt{\frac{x}{z}}\, c_2\, \frac{\cosh\sqrt{xz}\pm 1}{\sinh\sqrt{xz}}, && \text{(B.36)}\\
c_1 &= -\frac{\sinh^2\sqrt{xz}}{2xc_2}, \qquad h_1 = h_2 = 0, \qquad h_3 = \cosh\sqrt{xz}, && \text{(B.37)}
\end{aligned}
$$

with arbitrary h_0 and $c_2 \neq 0$. In this case, the realizations of de deformed Lie algebra (3.47), are given by

$$
\begin{aligned}
A &= \frac{(\cosh\sqrt{xz}\mp 1)}{2c_2\sqrt{xz}}\sinh\sqrt{xz}\, b - \sqrt{\frac{x}{z}}\, c_2\, \frac{\cosh\sqrt{xz}\pm 1}{\sinh\sqrt{xz}}\, b^\dagger && \text{(B.38)}\\
B &= I, && \text{(B.39)}\\
C &= -\frac{\sinh^2\sqrt{xz}}{2xc_2}\, b + c_2\, b^\dagger, && \text{(B.40)}\\
H &= h_0\, I + \cosh\sqrt{xz}\, b^\dagger b. && \text{(B.41)}
\end{aligned}
$$

Choosing the superior sign in (B.38), $h_0 = 0$ and

$$
c_2 = \frac{(\cosh\sqrt{xz}-1)}{2\sqrt{xz}}\sinh\sqrt{xz}, \tag{B.42}
$$

we get the realization (4.51–4.52), whereas if we chosse

$$
c_2 = \frac{\cosh\sqrt{xz}+1}{\sqrt{2x}}, \tag{B.43}
$$

we get the realization (4.53).

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